Transport properties of quasiparticles with fractional exclusion statistics

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(August 20, 1998)

Abstract

We consider the ballistic transport of quasiparticles with exclusion statistics through a 1D wire within the Landauer-Büttiker approach. We demonstrate that quasiparticle transport coefficients (electrical and heat conductance, as well as thermopower) are determined by the same general formulae as for particles with normal statistics. By applying the developed formalism to the ballistic transport of fractional charge it is shown that for a wire in contact to quasiparticles reservoirs the transport coefficients depend on the fractional charge. Specific features of resonant tunneling of quasiparticles are discussed.

05.30.-d,72.15.Jf,71.10.Pm

I. INTRODUCTION

Exclusion statistics was proposed[1] as a phenomenological description of excitations with mixed statistical properties (intermediate between fermions and bosons) in many-body systems. It is based on the assumption that the change of the number of available one-particle states in the system, Δd, at a given volume and at fixed boundary conditions depends *linearly* on the change of the number of quasiparticles ΔN: $\Delta d = -g\Delta N$. Here g is the statistical parameter which is assumed to be a rational number, with $g = 0$ and $g = 1$ corresponding to the normal bosonic and fermionic statistics, respectively. This generalized Pauli's principle results in a g-dependent statistical weight for the ideal gas of quasiparticles (usually referred as *exclusons*). The standard methods of statistical quantum mechanics allow one to derive the equilibrium distribution function of exclusons[2]

$$f_g(\varepsilon) = \left[y\left(\frac{\varepsilon - \mu_g}{T}\right) + g \right]^{-1}, \tag{1}$$

where T is the temperature (we consider units where $k_B = 1$), μ_g is the chemical potential of the ideal gas, and $y(x)$ is the real positive solution of the algebraic equation

$$y^g (1+y)^{1-g} = e^x. \tag{2}$$

Formally, exclusons can be regarded as composites of fermions and bosons since their thermodynamic potential can be decomposed into a sum of fermionic and bosonic potentials weighted by fractional coefficients,[3] namely,

$$\begin{aligned} \Omega_g &= -T \sum_j \ln \left[\frac{1 + (1-g) f_g(\varepsilon_j)}{1 - g f_g(\varepsilon_j)} \right] \\ &= g\Omega_f + (1-g)\Omega_b, \end{aligned} \tag{3}$$

where $\Omega_{g,f,b}$ denotes the grand thermodynamic potential for exclusons, fermions, and bosons, correspondingly, and $\mu_g = g\mu_f + (1-g)\mu_b$.

In spite of its phenomenological footing, exclusion statistics proved to be correct for quasiparticles in certain exactly solvable 1D models[4] and, more importantly, it is the statistics of recently observed fractionally charged excitations in fractional quantum Hall systems,[5] with the statistical parameter determined by the Landau level filling factor.

Already for two decades, fractional charge is a theoretically well-grounded concept, both in quantum field theory and in solid state physics (for reviews, see Ref. 6). It is known[7] that the requirement of gauge invariance forces the gas of fractionally charged particles at equilibrium to obey fractional statistics. From a phenomenological point of view, exclusion statistics is a reasonable candidate for the required anomalous statistics of fractionally charged quasiparticles. Thus, the study of thermodynamic and transport properties of *fractionally* charged exclusons can be more than simply an academic exercise.

In Ref. 8 the persistent current of a 1D ring of fractionally charged exclusons was calculated. It was shown that at low temperatures the fractional charge, masked by fractional statistics, does not show up in the Aharonov-Bohm oscillations. However, the "high-temperature" properties of the system do depend on fractional charge. Unfortunately, this

region is hardly accessible to present experiments since the oscillating current itself is exponentially small. The purpose of the present paper is to go beyond thermodynamics and consider transport properties of exclusons. We show within the Landauer-Büttiker formalism that the transport coefficients for exclusons are determined by the same general formulae as for the particles with normal statistics, provided the leads representing particle reservoirs are also described by exclusion statistics. In this case, even for an impurity-free 1D wire the transport coefficients depend on the statistical parameter. It is argued however that for an ideal wire the above dependence disappears when one considers "normal" particle reservoirs.

II. CONDUCTANCE

Let us consider an effectively infinite 1D wire of exclusons with a scattering potential described by the transmission probability $T_t(\varepsilon)$. We will assume at first that the wire is connected to reservoirs of *exclusons* in equilibrium at temperature T and chemical potential μ_g. In this case the generalization of Landauer's formula for conductance is straightforward: For exclusons with fractional charge $q_m = e/m$ (m is an integer) and statistical parameter $g = 1/m$ it reads

$$G_m = \frac{G_0}{m^2} \int_0^\infty d\varepsilon \; T_t(\varepsilon) \left[-\frac{\partial f_g(\varepsilon)}{\partial \varepsilon} \right], \tag{4}$$

where $G_0 = (2s+1)e^2/h$ is the conductance quantum (s is the spin of exclusons) and the distribution function is defined by Eqs. (1) and (2). To proceed we have to specify the properties of the scattering potential in the vicinity of the Fermi energy. If transport of exclusons through the barrier is nonresonant, the transmission probability at low temperatures ($T \ll \mu_g$) can be approximated by an energy independent factor $T_t(\mu_g)$. Then, Eq. (4) is reduced to a simple formula

$$G_m = \frac{G_0}{m} T_t(\mu_g). \tag{5}$$

In particular, for a perfect wire ($T_t = 1$) the conductance of fractionally charged exclusons $G_g = gG_0$ coincides with that for a homogeneous Luttinger liquid (LL) of integer charge particles with correlation parameter $g = 1/m$. However, in distinction from the LL case, exclusons are supposed to be noninteracting particles and therefore their scattering by a local potential can be described by a non-renormalized transmission probability $T_t(\mu_g)$. Thus the conductance is finite and temperature independent up to temperatures of the order of μ_g. Here a clarifying comment is needed. We are considering a gas of exclusons in a grand canonical ensemble, so μ_g is an input parameter, which is assumed to be positive. It means that for any $g \neq 0$ there exists a low temperature region where $T \ll \mu_g$. Yet, for $g \ll 1$ the chemical potential $\mu_g = g\mu_f - (1-g) \mid \mu_b \mid > 0$ can also be small and the only region realizable is the "high temperature" one $T \gg \mu_g$. In this case the conductance will be sensitive to the low energy dependence of the transmission coefficient. For a perfect wire ($T_t = 1$) the corresponding conductance will read $G_{m\gg1}(T \gg \mu_g) \simeq G_0/m \ln m$. In what follows we will assume that even for large (but finite) m the excluson chemical potential is still the largest energy scale in the problem.

The case of resonant tunneling of exclusons reveals far more interesting transport properties than the homogeneous case. To show why, we will approximate the resonant transmission coefficient $T_t(\varepsilon)$ at energies in the vicinity of the Fermi level ($\varepsilon \simeq \mu_g$) by the Breit-Wigner form

$$T_t^{(r)}(\varepsilon) = \frac{\Gamma^2}{(\varepsilon - \Delta)^2 + \Gamma^2}, \tag{6}$$

where Δ and Γ are the position and the width of the resonance level, respectively. The resonance tunneling implies that $\mu_g^{(r)} = \Delta$ and the corresponding conductance at $T = 0$ coincides (at it should be) with that for a perfect channel. At finite temperature the resonant conductance takes the form

$$G_m^{(r)}(T) = \frac{G_0}{m^2} \int_{-\infty}^{\infty} dx \frac{(\Gamma/T)^2}{x^2 + (\Gamma/T)^2} \left[-\frac{\partial f_g(x)}{\partial x} \right]. \tag{7}$$

The above expression determines the temperature dependence of the resonance peak height at $T \ll \mu_g$. Let us place the question - How does the peak height depend on the statistical parameter $g = 1/m$? If g is not small the resonance conductance is qualitatively the same as that for fermions ($m = 1$),

$$G_m^{(r)}(T) \simeq \frac{G_0}{m} \begin{cases} 1 - (A_m/m)(T/\Gamma)^2, & T \ll \Gamma \\ B_m(\Gamma/T), & T \gg \Gamma \end{cases}, \tag{8}$$

where

$$A_{1/g} = \int_{-\infty}^{\infty} dx\ x^2 \left[-\frac{\partial f_g(x)}{\partial x} \right] \tag{9}$$

and

$$B_{1/g} = \pi g \left[-\frac{\partial f_g(x)}{\partial x} \right]_{x=0}. \tag{10}$$

These coefficients can be evaluated as follows. With the help of Eq. (2) the derivative of the distribution function can be expressed in the form

$$f_g'(x) = -\frac{y(x)[1 + y(x)]}{[g + y(x)]^3} \tag{11}$$

Inserting this expression into Eq. (9) and changing the integration variable from x to y one finds for any $g \neq 0$ that the coefficient $A_{1/g}$ takes the same value as for fermions,

$$A_{1/g} = \int_0^{\infty} dy \frac{[g \ln y + (1 - g) \ln(1 + y)]^2}{(g + y)^2} = \frac{\pi^2}{3}. \tag{12}$$

This is a remarkable fact - the coefficient $A_{1/g}$ does not depend on the statistical parameter. In the next section it will be shown that this very coefficient determines the heat conductance

of exclusons. The numerical coefficient $B_{1/g}$ on the other hand, does depend on g and takes the values: $B_1 = \pi/4$, $B_2 = 4\pi/5\sqrt{5}$, ..., $B_{m\gg1} \sim m/\ln^2 m$.

For decreasing values of the statistical parameter the temperature dependence of the resonance conductance acquires new qualitative features. Namely, the crossover from low-T to high-T behavior is transformed into a wide region $\Gamma \ll T \ll m\Gamma$ where $G_m(T)$ for $m \gg 1$ depends almost *linearly* on temperature (see Fig.1). To study this region analytically it is tempting to use the bosonic limit. By expanding the distribution function of exclusons in the vicinity of the bosonic statistics ($g = 0$) it is easy to show that first two terms of $1/m$-expansion of resonant conductance, Eq. (7), take the form

$$G^{(r)}_{m\gg1}(T) \simeq \frac{G_0}{m}\left\{1 - \frac{1}{m}\left[\gamma\psi'(\gamma) - \frac{1}{2\gamma} - 1\right]\right\}, \tag{13}$$

where $\gamma \equiv \Gamma/2\pi T$ and $\psi' \equiv d^2 \ln\Gamma(x)/dx^2$ (here $\Gamma(x)$ is the gamma function[9]). For finite m the above expression holds formally until the term in the square brackets is much smaller than m. This restricts the temperature region where Eq. (13) could be at least qualitatively correct to temperatures $T \ll \Gamma_m \equiv m\Gamma/\pi$. According to Eq. (13), at low temperatures ($T < \Gamma$) the resonance conductance is still determined by the low-T asymptotics of Eq. (8). However, for $m \gg 1$ there is a wide temperature region $\Gamma < T \ll \Gamma_m$ where conductance depends *linearly* on temperature, $G^{(r)}_{m\gg1}(T) \simeq (G_0/m)(1-T/\Gamma_m)$. It is only at temperatures $T > \Gamma_m \gg \Gamma$ that the resonance conductance switches to the ordinary high-T asymptotics. Notice that Eq. (13) is not the true asymptotic expansion because all higher terms in $1/m$ are represented by divergent integrals. The formula is not quantitatively reliable and can be regarded at most as an estimate. We presented it only to show how the intermediate linear dependence on temperature appears in the crossover region of Eq. (8). It is worth noticing that the widening of the crossover region is a specific feature of the exclusion statistics for resonant tunneling. The above discussed linear-T regime is pronounced only for very small values of statistical parameter g.

The considerations above dealt with the transport of fractionally charged quasiparticles through a wire connected to reservoirs containing the particles of the same charge and statistics. Although theoretically conceivable, such situation is difficult to realize experimentally. A more natural experimental setup would be a 1D system of exclusons connected to leads containing noninteracting electron. Does the conductance of a perfect "excluson wire" connected to electron reservoirs depend on statistical parameter g? To solve this problem honestly one needs to introduce a microscopical model of fractional charge transport. In the phenomenological approach developed above it is reasonable to consider the case when the "excluson wire" is a metallic state characterized by a Fermi energy $\varepsilon_F^{(ex)}$. When an electron passes through the (presumably perfect) boundary to the "excluson wire" it is converted into m particles with fractional charge $q_m = e/m$. Therefore, the fractional charge in the expression for the quasiparticle current will be canceled by the extra factor m in the density of exclusons. At low temperatures and provided that there is no backscattering on the boundaries, a voltage applied to the electron reservoirs will result in a current through the system which will not depend on the fractional charge. Formally, the above statement can be proved by mapping the excluson model onto a LL model[10] (the mapping is exact at $T = 0$ and for the impurity-free case) and by recalling that the conductance of a LL constriction connecting to reservoirs of noninteracting electrons is $G_0 = (2s + 1)e^2/h$.[11]

III. THERMOELECTRIC EFFECTS

In the previous section we showed that the Landauer's formula for conductance can be applied to particles with exclusion statistics. Could we expand this claim to all transport coefficients? Since this is not evident from a general point-of-view, we at first derive the expressions for heat conductance $K_g(T)$ and thermoelectric cross coefficients $\Lambda_g(T)$ using the approach developed in Ref. 12.

Knowing the explicit expression for the grand thermodynamic potential of exclusons, Eq. (3), it is easy to show that the entropy of the ideal gas of exclusons takes the form

$$\begin{aligned} S_g = -\sum_j \{ & f_g(\varepsilon_j)\ln f_g(\varepsilon_j) + [1-gf_g(\varepsilon_j)]\ln[1-gf_g(\varepsilon_j)] \\ & -[1+(1-g)f_g(\varepsilon_j)]\ln[1+(1-g)f_g(\varepsilon_j)]\}\,, \end{aligned} \tag{14}$$

where $f_g(\varepsilon_j)$ is the distribution function of exclusons obtained from Eqs .(1) and (2). In the limiting cases of $g=1$ and $g=0$, Eq. (14) transforms into a well-known formula for the entropy of the ideal fermion and boson gases, respectively. With the entropy of exclusons at hand one can (following Ref. 12) perform all the transformations leading to the expressions for the heat conductance and the coefficient $\Lambda_g(T)$. Using Eqs. (1) and (2) one can show that the desired quantities are determined by the same general formulae as for ordinary particles,[12]

$$K_g(T) = \frac{(2s+1)}{h}\int_0^\infty d\varepsilon \frac{(\varepsilon-\mu_g)^2}{T} T_t(\varepsilon)\left[-\frac{\partial f_g(\varepsilon)}{\partial \varepsilon}\right]. \tag{15}$$

A similar expression [but linear on $(\varepsilon-\mu_g)$] stands for the thermoelectric coefficient Λ_g.

The analysis of Eq. (15) is straightforward. At low temperatures, $T \ll \mu_g$, one can replace the transmission probability by the energy-independent factor $T_t(\mu_g)$. Then, the heat conductance reads

$$K_g(T) = \frac{1}{h} T A_{1/g} T_t(\mu_g), \tag{16}$$

where the numerical factor $A_{1/g}$ is defined by Eqs. (9) and (12). Since the above coefficient does not depend on the statistical parameter, the low-T heat transport of exclusons is determined by the same formulae as for free fermions. Analogously, the thermoelectric coefficient of exclusons in the limit $\mu_g/T \to \infty$ vanishes irrespective of the value of g,

$$\Lambda_g(T) \propto \int_0^\infty dy \frac{g\ln y + (1-g)\ln(1+y)}{(g+y)^2} \equiv 0. \tag{17}$$

We remark that for bosonic statistics the limit $g \to 0$ in Eq. (15) should be taken before $T \to 0$. For the case of neutral bosons ($g=0, \mu_b=0$) the resulting heat conductance is sensitive to the energy dependence of the transmission probability at low energies. For a perfect channel ($T_t = 1$) and spinless particles, we obtain

$$K_{g\to 0}(T, \mu_g \to 0) = \frac{\pi^2}{3}\frac{1}{h}T, \tag{18}$$

which again coincides exactly with that for fermions.

An important characteristic of metallic systems is the ratio between the heat and electric conductances, $L(T) = K(T)/TG(T)$, known as the Lorentz number. [For Fermi liquids, $L_0 = (\pi^2/3)(1/e)^2$]. According to Eqs. (5) and (16) this quantity for exclusons, though temperature independent for $T \ll \mu_g$, depends on the statistical parameter $L_g = L_0/g$. This result coincides exactly with that for an infinite Luttinger liquid.[13]

ACKNOWLEDGMENTS

This work was supported by the Brazilian Agency CAPES. I.K. thanks the Physics Department at PUC-Rio (Rio de Janeiro) for the hospitality.

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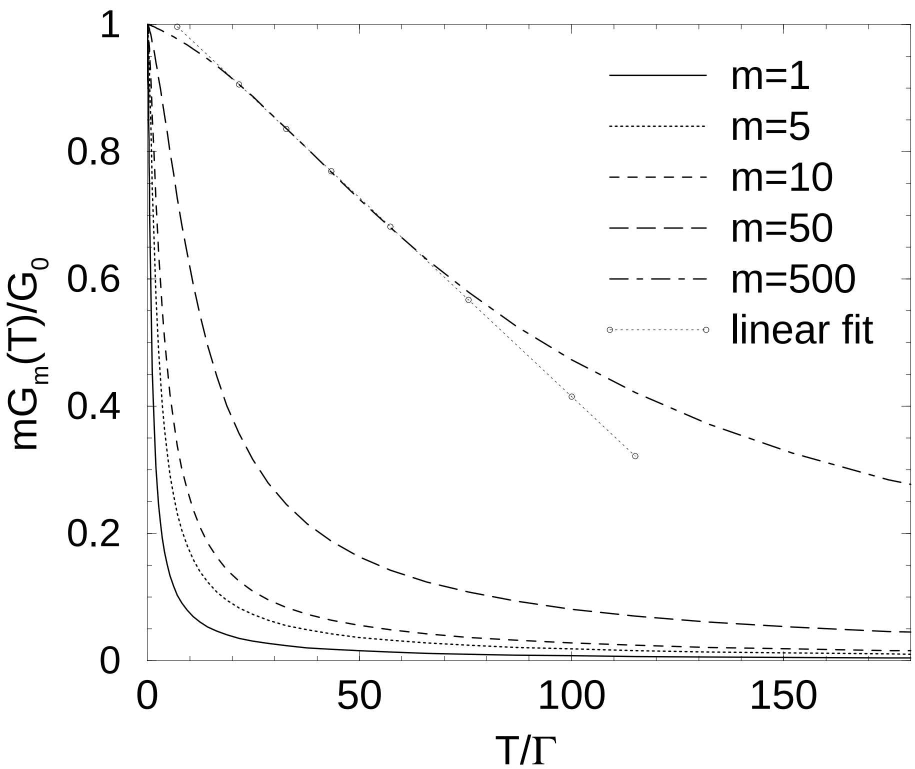

FIG. 1. Resonant conductance as a function of temperature for different values of the statistical parameter $g = 1/m$ [Eq. (7)]. The straight dot-dashed line is a linear fit to the $m = 500$ curve for $T < \Gamma_m$